Int. J. Solids Structures Vol. 25, No. 4, pp. 339–355, 1989
Printed in Great Britain.

0020–7683/89 \$3.00 + .00

A HIGHER-ORDER THEORY OF HYGROTHERMAL BEHAVIOR OF LAMINATED COMPOSITE SHELLS

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(*Received* 6 *April* 1988)

Abstract—A higher-order shell theory, which includes the effects of transverse shear and transverse normal strains, is developed for describing the behavior of composite shells. The equations, applicable to laminated composite shells of arbitrary shape with arbitrary temperature and moisture distributions, are established in tensor notation without reference to any particular coordinate system. These equations are then written in terms of components with respect to an orthogonal curvilinear coordinate system.

INTRODUCTION

The thermal properties, as well as their high specific stiffness and strength, have made fiber-reinforced composites ideal for many aerospace applications where deformations induced by temperature or moisture must be minimized. One such application is the support structure of space telescope mirrors in which many of the structural elements, including the mirrors themselves, are shells—essentially two-dimensional, thin, curved elements.

To utilize fully composite materials in such structures, a method to predict the deformation of these elements due to changes in temperature and moisture concentration must be available. The significance of the problem has led to many studies of the hygrothermal behavior of laminated composite shells. Most of the previous investigators were concerned with circular cylinders and spheres, and analyzed the problem via elasticity theory, by considering only one or two dimensions (e.g. see summaries by Takeuti and Naotake (1978) and by Hyer *et al.* (1986)). Three-dimensional analyses have been applied to fiber-reinforced laminated composite circular cylinders via shell theory by Stavsky and Smolash (1970), Pao (1972), Whitney (1971), Whitney and Sun (1974), Padovan and Lestingi (1980), and Hsu *et al.* (1981). In all but one of these analyses, either the effects of transverse normal (ε_{33}) or transverse shear strains (ε_{13} and ε_{23}) were neglected, the apparent exception being Whitney and Sun's (1974) study of hygrothermal deformations of circular cylinders. (Subscripts 1 and 2 denote directions parallel to the midsurface and subscript 3 denotes the direction perpendicular to the midsurface, Fig. 1.)

The effects of transverse normal strain (or thickening strain) are important in laminated composites, because, when heated, such materials tend to expand more in the direction perpendicular to the plane of the laminate than in the directions parallel to this plane. The importance of this "thickening" effect was discussed by Daugherty *et al.* (1971) in a

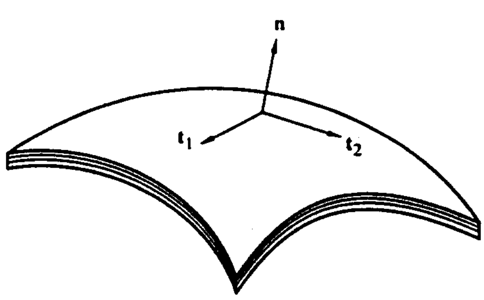

Fig. 1. Description of the shell.

technical note in which they developed governing equations for the free thermal expansion of homogeneous, orthotropic shells, and by Whitney (1971) and Whitney and Sun (1974) in their papers on composite circular cylinders. The importance of including transverse shear strains in the analysis of hygrothermal deformations was pointed out by Whitney and Sun (1974), Padovan and Lestingi (1980), and Hsu *et al.* (1981).

Thus, it is well recognized that, to achieve good accuracy, both the transverse normal strain and transverse shear strains must be included in the calculation of hygrothermal stresses and strains of composite shells. Indeed, these strains have already been included in elasticity and shell theory solutions of composite cylinders. However, corresponding analyses, taking into account both the normal and transverse strains, have not yet been developed for composite shells of general shape. Therefore, the first objective of this investigation was to derive the governing equations, which include the effects of transverse normal and transverse shear strains for laminated composite shells of arbitrary shape subjected to arbitrary moisture and temperature changes. These results are presented in this paper. The second objective was to study hygrothermal deformations of axisymmetric laminated composite shells with and without a sandwich core. These results are described in Doxsee and Springer (1989a,b).

PROBLEM STATEMENT

The deformation of a shell subjected to changes in temperature and moisture concentration is desired. The shell has a uniform thickness which is much smaller than the shell's radii of curvature (Fig. 1). The shell may be composed of a single material or several different materials bonded together in layers, each layer having a constant thickness. Each layer may be isotropic or orthotropic. The material properties are assumed to be linearly elastic and independent of stress, temperature, and moisture concentration. A consistent combination of displacements, forces, and moments are specified along the edges of the shell.

Initially, each point of the shell is at some arbitrary, but known, temperature, and the material at each point has absorbed a certain known amount of moisture. The shell is introduced to a new environment which causes known changes in the temperature and in the moisture concentration. The changes in turn induce internal stresses in and deformation of the shell. The displacement of each point of the shell is taken to be small compared to the thickness.

The following problem is addressed: given the initial geometry of the shell, its material properties, the prescribed edge forces and displacements, and the temperature and moisture concentration changes at every point of the shell, the displacements and stresses at every point of the shell are required. For this problem, mathematical models of the system have been obtained which are valid for generally shaped shells having arbitrary temperature and moisture distributions.

GOVERNING EQUATIONS

In this section a higher-order theory of the hygrothermal behavior of composite shells is developed. This shell theory includes both transverse normal and transverse shear strains and accounts for the coupling between the transverse normal and the in-plane strains. The governing equations are developed in tensor notation without reference to any particular coordinate system. It is noted that some of the derivations are very long and, hence, could not be included here. Readers interested in further details of the analysis are referred to the thesis by Doxsee (1988). The governing equations written in terms of components with respect to an orthogonal curvilinear coordinate system are given below.

As was done in many previous plate and shell theories, the variation of displacements (U_1, U_2, U_3) through the thickness of the shell is approximated as a polynomial in the

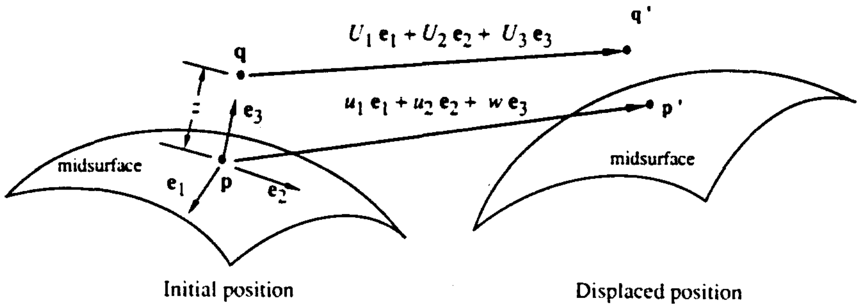

Fig. 2. Displacement approximation.

normal coordinate z (Fig. 2)

$$
\begin{aligned}
U_1 &= u_1 + z\,{}_1\beta_1 + z^2\,{}_2\beta_1 + z^3\,{}_3\beta_1 + \cdots \\
U_2 &= u_2 + z\,{}_1\beta_2 + z^2\,{}_2\beta_2 + z^3\,{}_3\beta_2 + \cdots \\
U_3 &= w + z\,{}_1\eta + z^2\,{}_2\eta + z^3\,{}_3\eta + \cdots
\end{aligned} \tag{1}
$$

where U_1 and U_2 are the displacements of a point **q**† in directions parallel to the midsurface, and U_3 the displacement of the point in the direction perpendicular to the midsurface.‡ The corresponding displacements of point **p** on the midsurface closest to **q** are denoted u_1, u_2, and w. The parameters ${}_1\beta_1$ and ${}_1\beta_2$ are midsurface "rotations". These and the other β's indicate the variations of in-plane displacements through the thickness of the shell. The η's determine the "stretching of the normals" and the transverse normal strain ε_{33}, since

$$
\varepsilon_{33} = \frac{dU_3}{dz} = {}_1\eta + 2z\,{}_2\eta + 3z^2\,{}_3\eta + \cdots. \tag{2}
$$

In order to account for transverse normal strain, the shell theory's displacement approximation must include η's.

GEOMETRY AND DEFINITION

A shell of constant thickness t is considered (Fig. 3), and the points of the shell and its boundary are denoted by $\mathscr{B}$ and $\partial\mathscr{B}$, respectively. The boundary of the shell is the union of the upper surface, the lower surface, and the edge faces $\mathscr{F}$. The set of points lying halfway between the upper and lower surfaces is called the midsurface and is denoted $\mathscr{S}$. The outward unit vector normal to $\partial\mathscr{B}$ is denoted **ν** and the intersection of $\mathscr{S}$ and $\mathscr{F}$ is denoted $\partial\mathscr{S}$.

Let **q** be any point in the shell. The point of the midsurface closest to **q** is denoted **p** and is related to **q** via

$$
\mathbf{q} = \mathbf{p} + z\mathbf{n}(\mathbf{p}) \tag{3}
$$

where z is the distance between points **p** and **q**, and **n(p)** is the unit vector normal to the midsurface at **p** (Fig. 4).

Any vector **v** defined at a point **q** may be decomposed into two vectors, one vector parallel to the midsurface at **p**, called the intrinsic part, and one vector normal to the midsurface at **p**, called the extrinsic part (Fig. 4). The intrinsic part will be denoted $\overset{i}{\mathbf{v}}$ and

† Boldface indicates that the quantity is either a point in space or a vector.

‡ Pre-subscripts are employed to differentiate different but similar quantities. Post-subscripts indicate the components of vectors and tensors.

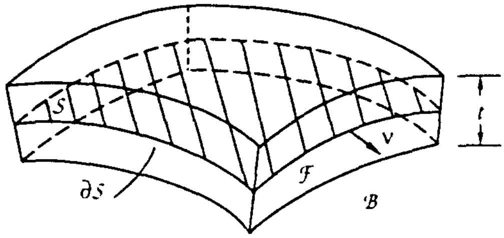

Fig. 3. Part of a shell $\mathscr{A}$.

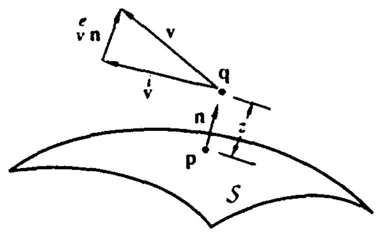

Fig. 4. Intrinsic $\overset{i}{\mathbf{v}}$ and extrinsic $\overset{e}{v}\mathbf{n}$ parts of a vector $\mathbf{v}$.

the extrinsic part will be denoted $\overset{e}{v}\mathbf{n}$ so that $\overset{i}{\mathbf{v}}$ is a vector and $\overset{e}{v}$ a scalar. Then the decomposition is written

$$\mathbf{v} = \overset{i}{\mathbf{v}} + \overset{e}{v}\mathbf{n}. \tag{4}$$

Second-rank tensors are decomposed as follows. Suppose a second-rank tensor $\mathbf{T}$ is the sum of dyads

$$\mathbf{T} = \mathbf{u} \otimes \mathbf{v} + \mathbf{t} \otimes \mathbf{w} \tag{5}$$

where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{t}$, and $\mathbf{w}$ are vectors and where $\otimes$ is the tensor product operator (Gurtin, 1981). Then $\mathbf{T}$ is decomposed as

$$\begin{aligned} \mathbf{T} &= (\overset{i}{\mathbf{u}} + \overset{e}{u}\mathbf{n}) \otimes (\overset{i}{\mathbf{v}} + \overset{e}{v}\mathbf{n}) + (\overset{i}{\mathbf{t}} + \overset{e}{t}\mathbf{n}) \otimes (\overset{i}{\mathbf{w}} + \overset{e}{w}\mathbf{n}) \\ &= \overset{ii}{\mathbf{T}} + \overset{ie}{\mathbf{T}} \otimes \mathbf{n} + \mathbf{n} \otimes \overset{ei}{\mathbf{T}} + \overset{ee}{T}(\mathbf{n} \otimes \mathbf{n}) \end{aligned} \tag{6}$$

where

$$\overset{ii}{\mathbf{T}} = \overset{i}{\mathbf{u}} \otimes \overset{i}{\mathbf{v}} + \overset{i}{\mathbf{t}} \otimes \overset{i}{\mathbf{w}}, \quad \overset{ee}{T} = \overset{e}{u}\overset{e}{v} + \overset{e}{t}\overset{e}{w}$$

$$\overset{ie}{\mathbf{T}} = \overset{e}{v}\overset{i}{\mathbf{u}} + \overset{e}{w}\overset{i}{\mathbf{t}}, \quad \overset{ei}{\mathbf{T}} = \overset{e}{u}\overset{i}{\mathbf{v}} + \overset{e}{t}\overset{i}{\mathbf{w}}.$$

The number of superscripts above $\mathbf{T}$ is equal to the rank of $\mathbf{T}$, and each superscript e reduces the rank by one. Hence, $\overset{ii}{\mathbf{T}}$ is a second-rank tensor, $\overset{ie}{\mathbf{T}}$ and $\overset{ei}{\mathbf{T}}$ are vectors, and $\overset{ee}{T}$ is a scalar. Since any second-rank tensor may be written as a sum of dyads, the above results generalize for all second-rank tensors. Analogous results hold for tensors of higher rank.

Two different double-dot products ($:$ and $\cdot\cdot$) will be used in deriving the governing equations. They are defined as follows. Let $\mathbf{C} = \mathbf{t} \otimes \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}$ be a fourth-rank tensor and let $\mathbf{T} = \mathbf{a} \otimes \mathbf{b}$ be a second-rank tensor. Then the two double-dot products of $\mathbf{C}$ with $\mathbf{T}$ are

$$\begin{aligned} \mathbf{C} : \mathbf{T} &= (\mathbf{t} \otimes \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}) : (\mathbf{a} \otimes \mathbf{b}) = (\mathbf{w} \cdot \mathbf{a})(\mathbf{t} \cdot \mathbf{b})\mathbf{u} \otimes \mathbf{v} \\ \mathbf{C} \cdot\cdot\, \mathbf{T} &= (\mathbf{t} \otimes \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}) \cdot\cdot\, (\mathbf{a} \otimes \mathbf{b}) = (\mathbf{w} \cdot \mathbf{a})(\mathbf{v} \cdot \mathbf{b})\mathbf{t} \otimes \mathbf{u}. \end{aligned} \tag{7}$$

The double-dot product (:) forms dot products between the pair of "inside" vectors ($\mathbf{w}\cdot\mathbf{a}$) and the pair of "outside" vectors ($\mathbf{t}\cdot\mathbf{b}$), while the double-dot product ($\cdot\cdot$) forms dot products between the two pairs of inside vectors. These definitions of the double-dot product generalize to tensors of other ranks.

KINEMATICS

Let $\mathbf{q}$ be a point in the shell and let $\mathbf{p}$ be its projection onto the midsurface as in eqn (3). The displacement of $\mathbf{q}$, denoted by $\mathbf{U}(\mathbf{q})$, is approximated by

$$\mathbf{U}(\mathbf{q}) = \mathbf{v}(\mathbf{p}) + \sum_{m=1}^{N} z^m \, {}_m\boldsymbol{\delta}(\mathbf{p}) \tag{8}$$

where $\mathbf{v}(\mathbf{p})$ is the displacement of $\mathbf{p}$; the second term of the sum ($z\,{}_1\boldsymbol{\delta}$) is the linear variation of displacement through the thickness; ($z^2\,{}_2\boldsymbol{\delta}$) is the quadratic variation of displacement through the thickness, and so on. The number N represents the "order" of the displacement approximation. The greater N is, the better eqn (8) approximates the actual deformation of the shell. Figure 5 shows a cross-sectional view of a shell in its initial and displaced positions. The point $\mathbf{q}$ and its projection $\mathbf{p}$ are displaced to the points $\mathbf{q}'$ and $\mathbf{p}'$, respectively.

As described above, the vectors $\mathbf{v}$ and ${}_m\boldsymbol{\delta}$ can be divided into vectors parallel and normal to the midsurface as

$$\mathbf{v} = \mathbf{u} + w\mathbf{n}, \quad {}_m\boldsymbol{\delta} = {}_m\boldsymbol{\beta} + {}_m\eta\mathbf{n}, \quad m = 1, 2, \ldots, N \tag{9}$$

where $\mathbf{u} = \overset{i}{\mathbf{v}}$ and ${}_m\boldsymbol{\beta} = {}_m\overset{i}{\boldsymbol{\delta}}$ are the vector components of $\mathbf{v}$ and ${}_m\boldsymbol{\delta}$ parallel to the midsurface (the intrinsic parts), and $w = \overset{e}{v}$ and ${}_m\eta = {}_m\overset{e}{\delta}$ are the magnitudes of the vector components normal to the surface (the extrinsic parts). The quantities $\mathbf{u}$, w, ${}_m\boldsymbol{\beta}$, and ${}_m\eta$ are the displacement measures of the shell theory. Taken together, eqns (8) and (9) are the vector representation of eqns (1).

Some of the $\boldsymbol{\beta}$'s and η's may be assumed to be zero. For future reference, N_β is defined to be the highest order of non-zero $\boldsymbol{\beta}$'s (i.e. ${}_m\boldsymbol{\beta} = \mathbf{0}$ for all $m > N_\beta$) and N_η is the highest order of non-zero η's (i.e. ${}_m\eta = 0$ for all $m > N_\eta$). N is then taken to be the greater of N_β and N_η.

The three-dimensional linear strain at a point $\mathbf{q}$ is given by (Gurtin, 1984)

$$\boldsymbol{\varepsilon}(\mathbf{q}) = \tfrac{1}{2}[\tilde{\nabla}\mathbf{U}(\mathbf{q}) + (\tilde{\nabla}\mathbf{U}(\mathbf{q}))^{\mathrm{T}}] \tag{10}$$

where the superscript T denotes the transpose, and $\tilde{\nabla}$ the gradient operator on three-dimensional space. This gradient operator is related to the gradient operator on the midsurface ∇ by (Steele, 1986; Doxsee, 1988)

$$\tilde{\nabla} = \boldsymbol{\mu}^{-1}\cdot\nabla + \mathbf{n}\otimes\frac{\partial}{\partial z}. \tag{11}$$

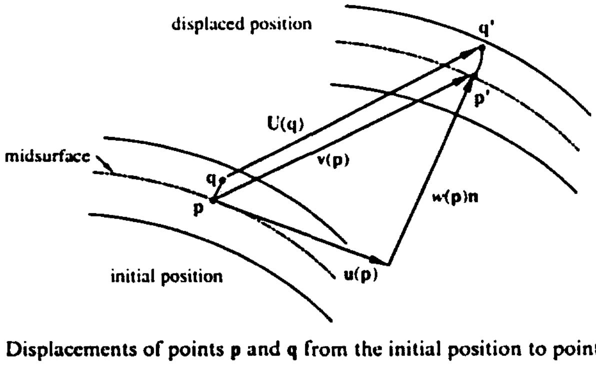

Fig. 5. Displacements of points p and q from the initial position to points p′ and q′.

In this equation, $\mu^{-1}(z)$ is the inverse of the tensor $\mu(z)$, which is defined by

$$\mu(z) = \mathbf{1} - z\mathbf{b} \tag{12}$$

where $\mathbf{1}$ is the identity tensor on the midsurface and $\mathbf{b} = -\nabla\mathbf{n}$ the curvature tensor of the midsurface. For plates, $\mathbf{b} = \mathbf{0}$.

By combining eqns (8)–(12), the strain at any point $\mathbf{q}$ can be expressed in terms of strain measures defined on the midsurface. The resulting expression is

$$\begin{aligned}\varepsilon(\mathbf{q}) = &\frac{1}{2}\left[\mu^{-1}\cdot\left(\gamma + \sum_{m=1}^{N} z^m\, {}_m\kappa\right) + \left(\gamma^{\mathrm{T}} + \sum_{m=1}^{N} z^m\, {}_m\kappa^{\mathrm{T}}\right)\cdot\mu^{-1}\right] \\ &+\frac{1}{2}\left[\mu^{-1}\cdot\left(\omega + \sum_{m=1}^{N} z^m\, {}_m\chi\right)\otimes\mathbf{n} + \mathbf{n}\otimes\left(\omega + \sum_{m=1}^{N} z^m\, {}_m\chi\right)\cdot\mu^{-1}\right] \\ &+\left(\rho + \sum_{m=1}^{N} z^m\, {}_m\lambda\right)\mathbf{n}\otimes\mathbf{n}\end{aligned} \tag{13}$$

where γ, ${}_m\kappa$, ω, ${}_m\chi$, ρ, and ${}_m\lambda$ are the strain measures. These strain measures are defined in terms of the displacement measures as

$$\begin{aligned} &\gamma = \overset{\text{I}}{\nabla}\mathbf{u} - w\mathbf{b}, && {}_m\kappa = \overset{\text{I}}{\nabla}\, {}_m\beta - {}_m\eta\mathbf{b} \\ &\omega = {}_1\beta + \mathbf{u}\cdot\mathbf{b} + \nabla w, && {}_m\chi = (m+1)\, {}_{(m+1)}\beta + (1-m)\, {}_m\beta\cdot\mathbf{b} + \nabla\, {}_m\eta \\ &\rho = {}_1\eta, && {}_m\lambda = (m+1)\, {}_{(m+1)}\eta \end{aligned} \tag{14}$$

with

$${}_{(N+1)}\beta = \mathbf{0}, \quad {}_{(N+1)}\eta = 0$$

where $\overset{\text{I}}{\nabla}$ is defined to be the operator which gives the intrinsic part of the gradient on the midsurface. The strain measures γ and ${}_m\kappa$ are tensors of the second rank, ω and ${}_m\chi$ are vectors, and ρ and ${}_m\lambda$ are scalars. In eqn (13), the terms on the right-hand side are the intrinsic ("in-plane") part of the strain tensor (first line), the transverse shear strain (second line), and the transverse normal strain (last line).

EQUILIBRIUM EQUATIONS AND BOUNDARY CONDITIONS

The equilibrium equations and boundary conditions are derived via the principle of virtual work (Fung, 1965). Let $\mathscr{B}$ be a shell with tractions $\nu\cdot\hat{\sigma}$ prescribed along part of its boundary $\partial\mathscr{B}_\sigma \subset \mathscr{F}$ and displacements prescribed along the other part $\partial\mathscr{B}_U \subset \mathscr{F}$. (The symbol $\subset$ represents a subset.) The upper and lower surfaces are taken to be traction free. Let $\bar{\mathbf{U}}(\mathbf{q})$ be a virtual displacement of each point $\mathbf{q}\in\mathscr{B}$ such that $\bar{\mathbf{U}}$ is zero on $\partial\mathscr{B}_U$ and is arbitrary elsewhere. The principle of virtual work states that (Fung, 1965)

$$0 = \int_{\partial\mathscr{B}_\sigma} \nu\cdot\hat{\sigma}\cdot\bar{\mathbf{U}}\,\mathrm{d}a + \int_{\mathscr{B}} [-\sigma : (\nabla\bar{\mathbf{U}})^{\mathrm{T}} + \mathbf{f}\cdot\bar{\mathbf{U}}]\,\mathrm{d}v \tag{15}$$

where σ is the stress tensor, and $\mathbf{f}$ the body force vector.

Virtual displacement measures, $\bar{\mathbf{u}}$, $\bar{w}$, $\overline{{}_m\beta}$, and $\overline{{}_m\eta}$ are defined to be related to the virtual displacement $\bar{\mathbf{U}}$ via the analog of eqns (8) and (9) (i.e. eqns (8) and (9) hold with bars over the displacement and the displacement measures). Similarly, virtual strain measures $\bar{\gamma}$, $\bar{\omega}$, $\overline{{}_m\kappa}$, $\overline{{}_m\chi}$, and $\overline{{}_m\lambda}$ are defined in terms of the virtual displacement measures via eqns (14) with the actual strain measures and actual displacement measures in eqns (14) replaced by the virtual strain measures and virtual displacement measures, respectively. By combining these results with eqns (10) and (13), integrating eqn (15) through the thickness, and performing other algebraic manipulations, one obtains

$$0 = \int_{\partial\mathscr{S}_\sigma} \boldsymbol{\nu}\cdot\left[\hat{\mathbf{N}}\cdot\bar{\mathbf{u}}+\hat{\mathbf{Q}}\bar{w}+\sum_{m=1}^{N}(\widehat{{}_m\mathbf{M}}\cdot\overline{{}_m\boldsymbol{\beta}}+\widehat{{}_m\mathbf{S}}\,\overline{{}_m\eta})\right]\mathrm{d}s$$
$$+\int_{\mathscr{S}}\left\{-\left[\mathbf{N}:\bar{\boldsymbol{\gamma}}^{\mathrm{T}}+\mathbf{Q}\cdot\bar{\boldsymbol{\omega}}+P\bar{\rho}+\sum_{m=1}^{N}({}_m\mathbf{M}:\overline{{}_m\boldsymbol{\kappa}}^{\mathrm{T}}+{}_m\mathbf{S}\cdot\overline{{}_m\boldsymbol{\chi}}+{}_mT\,\overline{{}_m\lambda})\right]\right.$$
$$\left.+\mathbf{l}\cdot\mathbf{u}+qw+\sum_{m=1}^{N}({}_m\mathbf{m}\cdot{}_m\boldsymbol{\beta}+{}_ms\,{}_m\eta)\right\}\mathrm{d}a. \tag{16}$$

The new terms appearing in this equation are defined below. The first integral is a line integral along the intersection of $\partial\mathscr{S}$ and $\partial\mathscr{B}_\sigma$, which is denoted $\partial\mathscr{S}_\sigma$, and the second integral is a surface integral over $\mathscr{S}$. Also appearing in eqn (16) are the stress resultants

$$\mathbf{N} = \int_{-t/2}^{t/2}\mu\boldsymbol{\mu}^{-1}\cdot\overset{\mathrm{ii}}{\boldsymbol{\sigma}}\,\mathrm{d}z,\quad {}_m\mathbf{M} = \int_{-t/2}^{t/2}\mu\boldsymbol{\mu}^{-1}\cdot\overset{\mathrm{ii}}{\boldsymbol{\sigma}}\,z^m\,\mathrm{d}z$$
$$\mathbf{Q} = \int_{-t/2}^{t/2}\mu\boldsymbol{\mu}^{-1}\cdot\overset{\mathrm{ie}}{\boldsymbol{\sigma}}\,\mathrm{d}z,\quad {}_m\mathbf{S} = \int_{-t/2}^{t/2}\mu\boldsymbol{\mu}^{-1}\cdot\overset{\mathrm{ie}}{\boldsymbol{\sigma}}\,z^m\,\mathrm{d}z$$
$$P = \int_{-t/2}^{t/2}\mu\overset{\mathrm{ee}}{\sigma}\,\mathrm{d}z,\qquad {}_mT = \int_{-t/2}^{t/2}\mu\overset{\mathrm{ee}}{\sigma}\,z^m\,\mathrm{d}z \tag{17}$$

for $m = 1, 2, \ldots, N$, where z is the normal coordinate, and μ the determinant of $\boldsymbol{\mu}$ which was defined by eqn (12). $\mathbf{N}$ is called the membrane stress resultant, $\mathbf{Q}$ the first transverse shear stress resultant, ${}_1\mathbf{M}$ the first stress resultant moment, and P the first transverse normal stress resultant. ${}_m\mathbf{M}$, ${}_m\mathbf{S}$, and ${}_mT$ are called higher-order stress resultant moments, transverse shear stress resultants, and transverse normal stress resultants, respectively. $\mathbf{N}$ and ${}_m\mathbf{M}$ are tensors of the second rank, $\mathbf{Q}$ and ${}_m\mathbf{S}$ are vectors, and P and ${}_mT$ are scalars. The body force resultants are defined as

$$\mathbf{l} = \int_{-t/2}^{t/2}\mu\overset{\mathrm{i}}{\mathbf{f}}\,\mathrm{d}z,\quad {}_m\mathbf{m} = \int_{-t/2}^{t/2}\mu\overset{\mathrm{i}}{\mathbf{f}}\,z^m\,\mathrm{d}z$$
$$q = \int_{-t/2}^{t/2}\mu\overset{\mathrm{e}}{f}\,\mathrm{d}z,\quad {}_ms = \int_{-t/2}^{t/2}\mu\overset{\mathrm{e}}{f}\,z^m\,\mathrm{d}z. \tag{18}$$

Finally, the prescribed traction resultants $\boldsymbol{\nu}\cdot\hat{\mathbf{N}}$, $\boldsymbol{\nu}\cdot\hat{\mathbf{Q}}$, $\boldsymbol{\nu}\cdot\widehat{{}_m\mathbf{M}}$, $\boldsymbol{\nu}\cdot\widehat{{}_m\mathbf{S}}$ are defined in terms of the prescribed tractions via eqn (17) pre-dotted on both sides with $\boldsymbol{\nu}$ and with hats placed over the stress resultants and stresses.

By making use of kinematical relations for virtual displacement measures and virtual strain measures (the analog of eqns (14)), the virtual strain measures in eqn (16) are replaced by expressions containing the virtual displacement measures and the gradients of the virtual displacement measures. Then by applying the divergence theorem to the resulting expression, and by noting that the virtual displacement measures are arbitrary everywhere except where displacements are prescribed, equilibrium equations and boundary conditions are obtained. The equilibrium equations are

$$\mathbf{0} = \overset{\mathrm{i}}{\nabla}\cdot\mathbf{N}-\mathbf{b}\cdot\mathbf{Q}+\mathbf{l}$$
$$0 = \mathbf{N}:\mathbf{b}+\nabla\cdot\mathbf{Q}+q$$
$$\mathbf{0} = \overset{\mathrm{i}}{\nabla}\cdot{}_m\mathbf{M}-(1-m)\mathbf{b}\cdot{}_m\mathbf{S}-m\,{}_{(m-1)}\mathbf{S}+{}_m\mathbf{m},\quad {}_0\mathbf{S} = \mathbf{Q}$$
$$0 = {}_m\mathbf{M}:\mathbf{b}+\nabla\cdot{}_m\mathbf{S}-m\,{}_{(m-1)}T+{}_ms,\quad {}_0T = P \tag{19}$$

for $m = 1, 2, \ldots, N$, where $(\overset{i}{\nabla}\cdot)$ is an operator which is defined to give the intrinsic part of the divergence on $\mathscr{S}$. There is a one-to-one correspondence between each of the above equations and the displacement measures adopted in the initial kinematic assumption, eqn (8). Equation $(19)_1$ corresponds to $\mathbf{u}$; i.e. if $\mathbf{u}$ is taken as a displacement measure, then eqn $(19)_1$ must be satisfied. Similarly, eqn $(19)_2$ corresponds to w, eqn $(19)_3$ corresponds to ${}_m\boldsymbol{\beta}$, for $m = 1, 2, \ldots, N_\beta$, and eqn $(19)_4$ corresponds to ${}_m\eta$, for $m = 1, 2, \ldots, N_\eta$.

As boundary conditions one must prescribe at each point of $\partial\mathscr{S}$

$$\begin{aligned} &\text{either} && \boldsymbol{\nu}\cdot\mathbf{N} && \text{or} && \mathbf{u} \\ &\text{and either} && \boldsymbol{\nu}\cdot\mathbf{Q} && \text{or} && w \\ &\text{and either} && \boldsymbol{\nu}\cdot{}_m\mathbf{M} && \text{or} && {}_m\boldsymbol{\beta} \\ &\text{and either} && \boldsymbol{\nu}\cdot{}_m\mathbf{S} && \text{or} && {}_m\eta \end{aligned} \tag{20}$$

for $m = 1, 2, \ldots, N$.

CONSTITUTIVE EQUATIONS

For the types of linearly elastic materials considered here, the stress–strain relation is (Carlson, 1984; Tsai and Hahn, 1980)

$$\boldsymbol{\sigma} = \mathbf{C}\cdot\cdot\boldsymbol{\varepsilon} + {}^h\theta\,{}^h\boldsymbol{\Phi} + {}^t\theta\,{}^t\boldsymbol{\Phi} \tag{21}$$

where $\mathbf{C}$ is the elasticity tensor, ${}^h\theta$ the change in specific moisture concentration from some reference value, ${}^t\theta$ the change in temperature from some reference temperature, ${}^h\boldsymbol{\Phi}$ the stress–moisture tensor, and ${}^t\boldsymbol{\Phi}$ the stress–temperature tensor. These quantities are explained in greater detail below.

The materials under consideration are orthotropic and do not exhibit coupling between intrinsic stresses $\overset{ii}{\boldsymbol{\sigma}}$ and transverse shear strain $\overset{ei}{\boldsymbol{\varepsilon}}$, nor between transverse shear stress $\overset{ie}{\boldsymbol{\sigma}}$ and transverse normal strain $\overset{ee}{\boldsymbol{\varepsilon}}$, etc. Thus many of the components of the elasticity tensor such as $\overset{iiie}{\mathbf{C}}$ and $\overset{ieee}{\mathbf{C}}$ are equal to zero.

The change in specific moisture concentration is the difference between the current specific moisture concentration c and a reference specific moisture concentration c_i,

$${}^h\theta = c - c_i. \tag{22}$$

The specific moisture concentration at a point $\mathbf{q}$ is defined by (Tsai and Hahn, 1980)

$$c = \lim_{\Delta V\to 0} \frac{\text{mass of moisture in } \Delta V}{\text{mass of dry material in } \Delta V} \tag{23}$$

where ΔV is the volume of a set of points surrounding $\mathbf{q}$. The stress–moisture tensor ${}^h\boldsymbol{\Phi}$ is symmetric and is related to the moisture-swelling tensor ${}^h\boldsymbol{\alpha}$ via

$${}^h\boldsymbol{\Phi} = -\mathbf{C}\cdot\cdot{}^h\boldsymbol{\alpha}. \tag{24}$$

Similarly, the stress–temperature tensor ${}^t\boldsymbol{\Phi}$ is symmetric and is related to the thermal-expansion tensor ${}^t\boldsymbol{\alpha}$ via

$${}^t\boldsymbol{\Phi} = -\mathbf{C}\cdot\cdot{}^t\boldsymbol{\alpha}. \tag{25}$$

Since problems for which the changes in temperature and moisture concentration are small are being considered, the material properties $\mathbf{C}$, ${}^h\boldsymbol{\alpha}$, and ${}^t\boldsymbol{\alpha}$ are taken to be independent of temperature, moisture concentration, and stress.

The distributions of moisture concentration change and temperature change through the thickness are approximated by

$$
\begin{aligned}
{}^{h}\theta(\mathbf{q}) &\approx \sum_{m=0}^{N_\theta} z^m\, {}^{h}_{m}\theta(\mathbf{p}) \\
{}^{t}\theta(\mathbf{q}) &\approx \sum_{m=0}^{N_\theta} z^m\, {}^{t}_{m}\theta(\mathbf{p})
\end{aligned}
\tag{26}
$$

where ${}^{h}_{m}\theta$ and ${}^{t}_{m}\theta$ are the specific moisture concentration measures and temperature measures of the shell theory, respectively. These measures are chosen to make eqns (26) approximate the actual distribution of temperature and moisture concentration as closely as possible. N_θ is the order number of the hygrothermal distribution approximation. In general, the greater N_θ is, the better eqns (26) approximate the actual temperature distribution.

By taking account of the symmetries of $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, and $\mathbf{C}$, and the fact that the material under consideration is orthotropic, the stress–strain relation, eqn (21), may be written as

$$
\begin{aligned}
\overset{ii}{\boldsymbol{\sigma}} &= \overset{iiii}{\mathbf{C}}\cdot\cdot\overset{ii}{\boldsymbol{\varepsilon}} + \overset{iiee}{\mathbf{C}}\overset{ee}{\varepsilon} + {}^{h}\theta\,{}^{h}\overset{ii}{\boldsymbol{\Phi}} + {}^{t}\theta\,{}^{t}\overset{ii}{\boldsymbol{\Phi}} \\
\overset{ie}{\boldsymbol{\sigma}} &= 2\overset{ieie}{\mathbf{C}}\cdot\overset{ie}{\boldsymbol{\varepsilon}} \\
\overset{ee}{\sigma} &= \overset{eeii}{\mathbf{C}}\cdot\cdot\overset{ii}{\boldsymbol{\varepsilon}} + \overset{eeee}{C}\overset{ee}{\varepsilon} + {}^{h}\theta\,{}^{h}\overset{ee}{\Phi} + {}^{t}\theta\,{}^{t}\overset{ee}{\Phi}
\end{aligned}
\tag{27}
$$

where the i's and e's above the terms indicate intrinsic and extrinsic parts, respectively. By taking the intrinsic and extrinsic parts of eqn (13), substituting them into the appropriate places in eqns (27) and then substituting these equations into eqns (17), the constitutive equations of the shell theory are obtained

$$
\begin{aligned}
\mathbf{N} &= {}_0\mathbf{B}\cdot\cdot\boldsymbol{\gamma} + {}_0\mathbf{D}\rho + \sum_{j=1}^{N} ({}_j\mathbf{B}\cdot\cdot{}_j\boldsymbol{\kappa} + {}_j\mathbf{D}\,{}_j\lambda) + {}^{h}\mathbf{N} + {}^{t}\mathbf{N} \\
{}_m\mathbf{M} &= {}_m\mathbf{B}\cdot\cdot\boldsymbol{\gamma} + {}_m\mathbf{D}\rho + \sum_{j=1}^{N} ({}_{(j+m)}\mathbf{B}\cdot\cdot{}_{(j+m)}\boldsymbol{\kappa} + {}_{(j+m)}\mathbf{D}\,{}_{(j+m)}\lambda) + {}^{h}_{m}\mathbf{M} + {}^{t}_{m}\mathbf{M} \\
\mathbf{Q} &= {}_0\mathbf{G}\cdot\boldsymbol{\omega} + \sum_{j=1}^{N} {}_j\mathbf{G}\cdot{}_j\boldsymbol{\chi} \\
{}_m\mathbf{S} &= {}_m\mathbf{G}\cdot\boldsymbol{\omega} + \sum_{j=1}^{N} {}_{(j+m)}\mathbf{G}\cdot{}_{(j+m)}\boldsymbol{\chi} \\
P &= {}_0\mathbf{D}^{\mathrm{T}}\cdot\cdot\boldsymbol{\gamma} + {}_0F\rho + \sum_{j=1}^{N} ({}_j\mathbf{D}^{\mathrm{T}}\cdot\cdot{}_j\boldsymbol{\kappa} + {}_jF\,{}_j\lambda) + {}^{h}P + {}^{t}P \\
{}_mT &= {}_m\mathbf{D}^{\mathrm{T}}\cdot\cdot\boldsymbol{\gamma} + {}_mF\rho + \sum_{j=1}^{N} ({}_{(j+m)}\mathbf{D}^{\mathrm{T}}\cdot\cdot{}_{(j+m)}\boldsymbol{\kappa} + {}_{(j+m)}F\,{}_{(j+m)}\lambda) + {}^{h}_{m}T + {}^{t}_{m}T
\end{aligned}
\tag{28}
$$

for $m = 1, 2, \ldots, N$. The terms ${}_m\mathbf{B}$, ${}_m\mathbf{D}$, ${}_m\mathbf{G}$, and ${}_mF$ ($m = 1, 2, \ldots, N$) are the shell elasticities defined as

$$
\begin{aligned}
{}_m\mathbf{B} &= \int_{-t/2}^{t/2} \mu\boldsymbol{\mu}^{-1}\cdot\overset{iiii}{\mathbf{C}}\cdot\boldsymbol{\mu}^{-1} z^m\,\mathrm{d}z, &\quad {}_m\mathbf{D} &= \int_{-t/2}^{t/2} \mu\boldsymbol{\mu}^{-1}\cdot\overset{iiee}{\mathbf{C}}\, z^m\,\mathrm{d}z \\
{}_mF &= \int_{-t/2}^{t/2} \mu\overset{eeee}{C} z^m\,\mathrm{d}z, &\quad {}_m\mathbf{G} &= \int_{-t/2}^{t/2} \mu\boldsymbol{\mu}^{-1}\cdot\overset{ieie}{\mathbf{C}}\cdot\boldsymbol{\mu}^{-1} z^m\,\mathrm{d}z.
\end{aligned}
\tag{29}
$$

Each ${}_m\mathbf{B}$ is a fourth-rank tensor, each ${}_m\mathbf{D}$ and ${}_m\mathbf{G}$ a second-rank tensor, and each ${}_mF$ a scalar. Also appearing in eqns (28) are the stress–moisture resultants

$$ {}^{h}\mathbf{N} = \int_{-t/2}^{t/2} {}^{h}\theta\mu\boldsymbol{\mu}^{-1}\cdot{}^{h}\overset{ii}{\boldsymbol{\Phi}}\,\mathrm{d}z, \quad {}^{h}_{m}\mathbf{M} = \int_{-t/2}^{t/2} {}^{h}\theta\mu\boldsymbol{\mu}^{-1}\cdot{}^{h}\overset{ii}{\boldsymbol{\Phi}}\,z^{m}\,\mathrm{d}z $$

$$ {}^{h}P = \int_{-t/2}^{t/2} {}^{h}\theta\mu\,{}^{h}\overset{cc}{\Phi}\,\mathrm{d}z, \qquad {}^{h}_{m}T = \int_{-t/2}^{t/2} {}^{h}\theta\mu\,{}^{h}\overset{cc}{\Phi}\,z^{m}\,\mathrm{d}z \tag{30} $$

for $m = 1, 2, \ldots, N$. Similarly, the stress–temperature resultants ${}^{t}\mathbf{N}$, ${}^{t}_{m}\mathbf{M}$, ${}^{t}P$, and ${}^{h}_{m}T$ are defined via eqns (30) with superscript h replaced by superscript t.

For theories in which N_{β} and N_{η} are low numbers, it is necessary to modify some of the above definitions for the elasticity, stress–moisture, and stress–temperature tensors. For example, there are two sets of modifications required for a first-order transverse shear deformation theory (i.e. $N_{\beta} = 1$ and $N_{\eta} = 0$). First, since $N_{\eta} = 0$ the definitions of ${}_{m}\mathbf{B}$, ${}^{t}\mathbf{N}$, ${}^{h}\mathbf{N}$, ${}^{h}_{m}\mathbf{M}$, and ${}^{t}_{m}\mathbf{M}$ must be modified. Second, since $N_{\beta} = 1$, the definition of ${}_{0}\mathbf{G}$ must be modified.

(1) Taking $N_{\eta} = 0$ actually follows from the assumption that, since plates and shells are thin, the transverse normal stress is often negligible (i.e. $\overset{cc}{\sigma} = 0$). In this case, the transverse normal strain $\overset{cc}{\varepsilon}$ is calculated from eqn $(27)_3$, and thus ${}_{m}\eta$, ρ, and ${}_{m}\lambda$ are not independent kinematic variables of the theory. Then the definition of ${}_{m}\mathbf{B}$, ${}^{t}\mathbf{N}$, ${}^{h}\mathbf{N}$, ${}^{h}_{m}\mathbf{M}$, and ${}^{t}_{m}\mathbf{M}$ must be modified as follows. By solving eqn $(27)_3$ (with $\overset{cc}{\sigma} = 0$) for $\overset{cc}{\varepsilon}$ and substituting the result into eqn $(27)_1$ one obtains

$$ \overset{ii}{\boldsymbol{\sigma}} = \overset{iiii}{\tilde{\mathbf{C}}}\cdot\cdot\overset{ii}{\boldsymbol{\varepsilon}} + {}^{h}\theta\,{}^{h}\overset{ii}{\tilde{\boldsymbol{\Phi}}} + {}^{t}\theta\,{}^{t}\overset{ii}{\tilde{\boldsymbol{\Phi}}} \tag{31} $$

where

$$ \overset{iiii}{\tilde{\mathbf{C}}} = \overset{iiii}{\mathbf{C}} - \frac{\overset{iicc}{\mathbf{C}} \otimes \overset{iicc}{\mathbf{C}}}{\overset{cccc}{C}} $$

$$ {}^{h}\overset{ii}{\tilde{\boldsymbol{\Phi}}} = {}^{h}\overset{ii}{\boldsymbol{\Phi}} - {}^{h}\overset{cc}{\Phi}\,\frac{\overset{iicc}{\mathbf{C}}}{\overset{cccc}{C}} \qquad {}^{t}\overset{ii}{\tilde{\boldsymbol{\Phi}}} = {}^{t}\overset{ii}{\boldsymbol{\Phi}} - {}^{t}\overset{cc}{\Phi}\,\frac{\overset{iicc}{\mathbf{C}}}{\overset{cccc}{C}}. \tag{32} $$

$\overset{iiii}{\tilde{\mathbf{C}}}$, ${}^{h}\overset{ii}{\tilde{\boldsymbol{\Phi}}}$, ${}^{t}\overset{ii}{\tilde{\boldsymbol{\Phi}}}$ are called the reduced intrinsic elasticity, stress–moisture, and stress–temperature tensors, respectively. Then ${}_{m}\mathbf{B}$, ${}^{t}\mathbf{N}$, ${}^{h}\mathbf{N}$, ${}^{h}_{m}\mathbf{M}$, and ${}^{t}_{m}\mathbf{M}$ are defined by eqns $(29)_1$ and (30) with $\overset{iiii}{\mathbf{C}}$, ${}^{h}\overset{ii}{\boldsymbol{\Phi}}$, and ${}^{t}\overset{ii}{\boldsymbol{\Phi}}$ replaced by $\overset{iiii}{\tilde{\mathbf{C}}}$, ${}^{h}\overset{ii}{\tilde{\boldsymbol{\Phi}}}$, and ${}^{t}\overset{ii}{\tilde{\boldsymbol{\Phi}}}$, respectively.

(2) For static equilibrium, the transverse shear stress $\overset{ic}{\boldsymbol{\sigma}}$ and strain $\overset{ic}{\boldsymbol{\varepsilon}}$ distributions through the thickness are roughly parabolic for symmetric laminates (Pagano and Hatfield, 1972). It follows from eqns (13) and (14) that the kinematics of the shell (and plate) theory reflect this fact only if $N_{\beta} \geqslant 3$. Thus for "low order transverse shear deformation theories" (i.e. theories which do not include the Kirchoff–Love assumption and for which $N_{\beta} < 3$) the definition of the shell elasticity tensor ${}_{m}\mathbf{G}$ (eqn $(29)_4$) must be modified. Historically, this modification has been accomplished in one of two different ways.

(a) The right-hand side of eqn $(29)_4$ is multiplied by shear correction factors which are chosen to make eqn $(28)_3$ as accurate as possible for a specific problem. This technique was introduced by Mindlin (1951) for plates, and is closely related to Timoshenko's beam theory (Timoshenko, 1922). Many previous investigators have used this or a related technique.

(b) In addition to the kinematic assumptions (eqns (8) and (9))—or instead of these assumptions—one makes assumptions concerning the distribution of stresses and strains through the thickness of the shell. Then a variational principle is employed to derive the constitutive equations. This is the technique employed by Reissner (1945, 1947, 1972, 1979) and Naghdi (1957, 1963, 1984).

For a first-order shear deformation theory, method (b) outlined above is chosen and Naghdi's (1963) procedure adopted. Naghdi assumed that $\overset{ic}{\boldsymbol{\varepsilon}}$ and $\overset{ic}{\boldsymbol{\sigma}}$ vary quadratically through the thickness, and then employed the Hu–Washizu (Gurtin, 1984) variational principle to obtain

$$ {}_0\mathbf{G} = \left(\frac{5}{6}\right)\left(\frac{15}{8}\right)\int_{-t/2}^{t/2} \mu\mu^{-1}\cdot \overset{ieie}{\mathbf{C}}\cdot\mu^{-1}\left[1-\left(\frac{z}{h/2}\right)^2\right]^2 \mathrm{d}z. \tag{33} $$

For a homogeneous material, this definition is equivalent to employing a shear correction coefficient of 5/6.

The derivation of the governing equations of the shell theories under investigation is now complete. The governing equations form a set of linear, partial differential equations:

(i) displacement measure—strain measure (kinematic) relations (eqns (14)),
(ii) equilibrium equations (eqns (19)),
(iii) stress-resultant—strain measure (constitutive) relations (eqns (28)), and
(iv) subject to the boundary conditions (eqns (20)).

Once the solution to this set of equations has been obtained, one determines at any point in the shell the displacement from eqn (8), the strain from eqn (13), and the stress from eqn (27). The accuracy of these equations depends on the number of terms included in the initial displacement approximation (eqn (8)), and in the temperature and moisture distribution approximations (eqn (26)). The accuracy also depends on the approximations made when evaluating the constitutive coefficients (eqn (29)), as discussed in the following section.

GOVERNING EQUATIONS IN LINES OF CURVATURE COORDINATES

In order to obtain numerical solutions to the governing equations derived in the previous chapter, it is necessary to express these equations in component form. For convenience, a line of curvature coordinate system (x_1, x_2, z) is adopted associated with the midsurface (Fig. 6) (Kraus, 1967). Curves of constant x_1 coincide with curves of principal curvature $1/R_2$ of the midsurface, and curves of constant x_2 coincide with curves of principal curvature $1/R_1$. The square of the length of a differential line segment on the midsurface of the shell is given by

$$ (\mathrm{d}l)^2 = A_1^2(\mathrm{d}x_1)^2 + A_2^2(\mathrm{d}x_2)^2 \tag{34} $$

where A_1 and A_2 are scalars which are functions of position (x_1, x_2) on the midsurface. The four quantities A_1, A_2, R_1, and R_2 define the shape of the shell and are not independent (Kraus, 1967).

At each point (x_1, x_2) of the midsurface of the shell a set of basis unit vectors $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{n})$ is defined such that $\mathbf{t}_1$ is in the direction of increasing x_1, $\mathbf{t}_2$ is in the direction of increasing x_2, and $\mathbf{n} = \mathbf{t}_1 \times \mathbf{t}_2$ is normal to the midsurface. As above, z is the coordinate in the direction of $\mathbf{n}$.

The governing equations are listed in Tables 1–10 in terms of physical components in

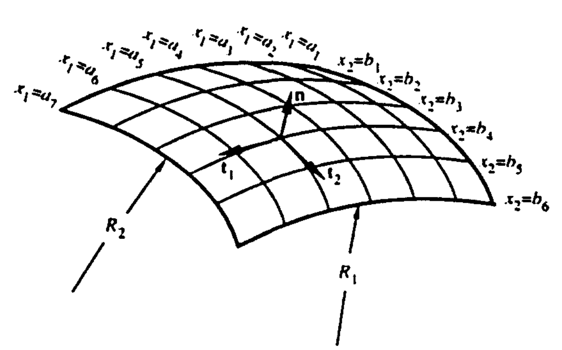

Fig. 6. Lines of curvature coordinates.

Table 1. Displacement in terms of displacement measures expressed in physical components in lines of curvature coordinates

$$U_\alpha(x_1,x_2,z) = u_\alpha(x_1,x_2) + \sum_{m=1}^{N_\beta} z^m\,{}_m\beta_\alpha(x_1,x_2)$$

$$U_3(x_1,x_2,z) = w(x_1,x_2) + \sum_{m=1}^{N_\eta} z^m\,{}_m\eta(x_1,x_2)$$

Subscript α takes on values 1 and 2.

Table 2. Strain in terms of strain measures expressed in physical components in lines of curvature coordinates

$$\varepsilon_{\alpha\beta} = \varepsilon_{\beta\alpha} = \frac{1}{2}\left[\frac{1}{1+z/R_\alpha}\left(\gamma_{\alpha\beta}+\sum_{m=1}^{N} z^m\,{}_m\kappa_{\alpha\beta}\right)+\frac{1}{1+z/R_\beta}\left(\gamma_{\beta\alpha}+\sum_{m=1}^{N} z^m\,{}_m\kappa_{\beta\alpha}\right)\right]$$

$$\varepsilon_{\alpha 3} = \varepsilon_{3\alpha} = \frac{1}{2}\left(\frac{1}{1+z/R_\alpha}\right)\left(\omega_\alpha+\sum_{m=1}^{N} z^m\,{}_m\chi_\alpha\right)$$

$$\varepsilon_{33} = \rho + \sum_{m=1}^{N} z^m\,{}_m\lambda$$

N is the greater of N_β, N_η. Subscripts α and β take on values 1 and 2 but there is no summation over repeated subscripts.

Table 3. Strain measures in terms of displacement measures expressed in physical components in lines of curvature coordinates

$$\gamma_{11} = \frac{1}{A_1}\left[u_{1,1}+\frac{A_{1,2}}{A_2}u_2\right]+\frac{w}{R_1}$$

$$\gamma_{22} = \frac{1}{A_2}\left[u_{2,2}+\frac{A_{2,1}}{A_1}u_1\right]+\frac{w}{R_2}$$

$$\gamma_{12} = \frac{1}{A_1}\left[u_{2,1}-\frac{A_{1,2}}{A_2}u_1\right]$$

$$\gamma_{21} = \frac{1}{A_2}\left[u_{1,2}-\frac{A_{2,1}}{A_1}u_2\right]$$

$$\omega_\alpha = {}_1\beta_\alpha - \frac{u_\alpha}{R_\alpha}+\frac{w_{,\alpha}}{A_\alpha}$$

$$\rho = {}_1\eta$$

$${}_m\kappa_{11} = \frac{1}{A_1}\left[{}_m\beta_{1,1}+\frac{A_{1,2}}{A_2}\,{}_m\beta_2\right]+\frac{{}_m\eta}{R_1}$$

$${}_m\kappa_{22} = \frac{1}{A_2}\left[{}_m\beta_{2,2}+\frac{A_{2,1}}{A_1}\,{}_m\beta_1\right]+\frac{{}_m\eta}{R_2}$$

$${}_m\kappa_{12} = \frac{1}{A_1}\left[{}_m\beta_{2,1}-\frac{A_{1,2}}{A_2}\,{}_m\beta_1\right]$$

$${}_m\kappa_{21} = \frac{1}{A_2}\left[{}_m\beta_{1,2}-\frac{A_{2,1}}{A_1}\,{}_m\beta_2\right]$$

$${}_m\chi_\alpha = (m+1)\,{}_{(m+1)}\beta_\alpha-(1-m)\frac{{}_m\beta_\alpha}{R_\alpha}+\frac{{}_m\eta_{,\alpha}}{A_\alpha}$$

$${}_m\lambda = (m+1)\,{}_{(m+1)}\eta$$

The comma notation $(\ )_{,\alpha}$ denotes differentiation with respect to the coordinate x_α. Subscript α takes on values 1 and 2 but there is no summation over repeated subscripts.

Table 4. Stress resultants in terms of stresses expressed in physical components in lines of curvature coordinates

$$N_{\alpha\beta} = \int_{-t/2}^{t/2} (1+z/\bar{R}_\alpha)\sigma_{\alpha\beta}\,\mathrm{d}z$$

$$Q_\alpha = \int_{-t/2}^{t/2} (1+z/\bar{R}_\alpha)\sigma_{\alpha 3}\,\mathrm{d}z$$

$$P = \int_{-t/2}^{t/2} (1+z/R_1)(1+z/R_2)\sigma_{33}\,\mathrm{d}z$$

$${}_mM_{\alpha\beta} = \int_{-t/2}^{t/2} (1+z/\bar{R}_\alpha)z^m\sigma_{\alpha\beta}\,\mathrm{d}z$$

$${}_mS_\alpha = \int_{-t/2}^{t/2} (1+z/\bar{R}_\alpha)z^m\sigma_{\alpha 3}\,\mathrm{d}z$$

$${}_mT = \int_{-t/2}^{t/2} (1+z/R_1)(1+z/R_2)z^m\sigma_{33}\,\mathrm{d}z$$

$\bar{R}_1 = R_2$ and $\bar{R}_2 = R_1$. Subscripts α and β take on values 1 and 2 but there is no summation over repeated subscripts.

Table 5. Body force resultants in terms of body forces expressed in physical components in lines of curvature coordinates

$$l_\alpha = \int_{-t/2}^{t/2} (1+z/R_1)(1+z/R_2)f_\alpha\,\mathrm{d}z$$

$$q = \int_{-t/2}^{t/2} (1+z/R_1)(1+z/R_2)f_3\,\mathrm{d}z$$

$${}_mm_\alpha = \int_{-t/2}^{t/2} (1+z/R_1)(1+z/R_2)f_\alpha z^m\,\mathrm{d}z$$

$${}_ms = \int_{-t/2}^{t/2} (1+z/R_1)(1+z/R_2)f_3 z^m\,\mathrm{d}z$$

Subscript α takes on values 1 and 2.

Table 6. Equilibrium equations expressed in physical components in lines of curvature coordinates

$$0 = \frac{1}{A_1A_2}[(A_2N_{11})_{,1} + (A_1N_{21})_{,2} + A_{1,2}N_{12} - A_{2,1}N_{22}] + \frac{Q_1}{R_1} + l_1$$

$$0 = \frac{1}{A_1A_2}[(A_1N_{22})_{,2} + (A_2N_{12})_{,1} + A_{2,1}N_{21} - A_{1,2}N_{11}] + \frac{Q_2}{R_2} + l_2$$

$$0 = \frac{1}{A_1A_2}[(A_2Q_1)_{,1} + (A_1Q_2)_{,2}] - \left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2}\right) + q$$

$$0 = \frac{1}{A_1A_2}[(A_2\,{}_mM_{11})_{,1} + (A_1\,{}_mM_{21})_{,2} + A_{1,2}\,{}_mM_{12} - A_{2,1}\,{}_mM_{22}] + (1-m)\frac{{}_mS_1}{R_1} - m\,{}_{(m-1)}S_1 + {}_mm_1$$

$$0 = \frac{1}{A_1A_2}[(A_1\,{}_mM_{22})_{,2} + (A_2\,{}_mM_{12})_{,1} + A_{2,1}\,{}_mM_{21} - A_{1,2}\,{}_mM_{11}] + (1-m)\frac{{}_mS_2}{R_2} - m\,{}_{(m-1)}S_2 + {}_mm_2$$

$$0 = \frac{1}{A_1A_2}[(A_2\,{}_mS_1)_{,1} + (A_1\,{}_mS_2)_{,2}] - \left(\frac{{}_mM_{11}}{R_1} + \frac{{}_mM_{22}}{R_2}\right) - m\,{}_{(m-1)}T + {}_ms$$

The comma notation $(\ )_{,\alpha}$ denotes differentiation with respect to the coordinate x_α.

Table 7. Boundary conditions expressed in physical components in lines of curvature coordinates

Must prescribe at each point of the boundary:

either	$N_{\nu\nu}$	or	u_ν
and either	$N_{\nu\lambda}$	or	u_λ
and either	Q_ν	or	w
and either	${}_mM_{\nu\nu}$	or	${}_m\beta_\nu$
and either	${}_mM_{\nu\lambda}$	or	${}_m\beta_\lambda$
and either	${}_mS_\nu$	or	${}_m\eta$

for $m = 1, 2, \ldots, N$. Subscript ν denotes the component in the direction normal to the boundary of the shell, and subscript λ denotes the component in the direction tangent to the boundary of the shell.

Table 8. Constitutive equations expressed in physical components in lines of curvature coordinates

$$N_{\alpha\beta} = {}_0B_{\alpha\beta\gamma\delta}\gamma_{\delta\gamma} + {}_0D_{\alpha\beta}\rho + \sum_{i=1}^{N} ({}_iB_{\alpha\beta\gamma\delta}\,{}_i\kappa_{\delta\gamma} + {}_iD_{\alpha\beta}\,{}_i\lambda) + {}^hN_{\alpha\beta} + {}^tN_{\alpha\beta}$$

$$Q_\alpha = {}_0G_{\alpha\beta}\omega_\beta + \sum_{i=1}^{N} {}_iG_{\alpha\beta}\,{}_i\chi_\beta$$

$$P = {}_0D_{\alpha\beta}\gamma_{\alpha\beta} + {}_0F\rho + \sum_{i=1}^{N} ({}_iD_{\alpha\beta}\,{}_i\kappa_{\alpha\beta} + {}_iF_i\lambda) + {}^hP + {}^tP$$

$${}_mM_{\alpha\beta} = {}_mB_{\alpha\beta\gamma\delta}\gamma_{\delta\gamma} + {}_mD_{\alpha\beta}\rho + \sum_{i=1}^{N} ({}_{(m+i)}B_{\alpha\beta\gamma\delta\,(m+i)}\kappa_{\delta\gamma} + {}_{(m+i)}D_{\alpha\beta\,(m+i)}\lambda) + {}^h_mM_{\alpha\beta} + {}^t_mM_{\alpha\beta}$$

$${}_mS_\alpha = {}_mG_{\alpha\beta}\omega_\beta + \sum_{i=1}^{N} {}_{(m+i)}G_{\alpha\beta\,(m+i)}\chi_\beta$$

$${}_mT = {}_mD_{\alpha\beta}\gamma_{\alpha\beta} + {}_mF\rho + \sum_{i=1}^{N} ({}_{(m+i)}D_{\alpha\beta\,(m+i)}\kappa_{\alpha\beta} + {}_{(m+i)}F_{(m+i)}\lambda) + {}^h_mT + {}^t_mT$$

Summation from 1 to 2 is implied by repeated Greek subscripts.

Table 9. Elasticities expressed in physical components in lines of curvature coordinates

$${}_mB_{\alpha\beta\gamma\delta} = {}_mB_{\gamma\delta\alpha\beta} = {}_mC_{\alpha\beta\gamma\delta} + J_{\alpha\delta\,(m+1)}C_{\alpha\beta\gamma\delta} + K_{\alpha\delta\,(m+2)}C_{\alpha\beta\gamma\delta} + \cdots$$

$${}_mD_{\alpha\beta} = {}_mC_{\alpha\beta33} + \frac{1}{\bar{R}_\alpha}\,{}_{(m+1)}C_{\alpha\beta33}$$

$${}_mG_{\alpha\beta} = {}_mG_{\beta\alpha} = {}_mC_{\alpha3\beta3} + J_{\alpha\beta\,(m+1)}C_{\alpha3\beta3} + K_{\alpha\beta\,(m+2)}C_{\alpha3\beta3} + \cdots$$

$${}_mF = {}_mC_{3333} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right){}_{(m+1)}C_{3333} + \frac{1}{R_1R_2}\,{}_{(m+2)}C_{3333}$$

where

$$J_{\alpha\delta} = \begin{cases} \dfrac{1}{R_2} - \dfrac{1}{R_1}, & \text{if } \alpha = \delta = 1 \\[2ex] \dfrac{1}{R_1} - \dfrac{1}{R_2}, & \text{if } \alpha = \delta = 2 \\[2ex] 0, & \text{otherwise} \end{cases}$$

$$K_{\alpha\delta} = \begin{cases} \dfrac{1}{R_1}\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right), & \text{if } \alpha = \delta = 1 \\[2ex] \dfrac{1}{R_2}\left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right), & \text{if } \alpha = \delta = 2 \\[2ex] 0, & \text{otherwise.} \end{cases}$$

The definition of ${}_mC_{ijkl}$ is given in eqns (36). The Greek subscripts take on values 1 and 2 but there is no summation over repeated subscripts. Also $\bar{R}_1 = R_2$ and $\bar{R}_2 = R_1$.

Table 10. Stress–moisture resultants expressed in physical components in lines of curvature coordinates

$$ {}^hN_{\alpha\beta} = {}^h_0\theta\,{}^h_0\Phi_{\alpha\beta} + \left(\frac{1}{\bar{R}_\alpha}{}^h_0\theta + {}^h_1\theta\right){}^h_1\Phi_{\alpha\beta} + \left(\frac{1}{\bar{R}_\alpha}{}^h_1\theta + {}^h_2\theta\right){}^h_2\Phi_{\alpha\beta} + \frac{1}{\bar{R}_\alpha}{}^h_2\theta\,{}^h_3\Phi_{\alpha\beta} $$

$$ {}^h_mM_{\alpha\beta} = {}^h_0\theta\,{}^h_m\Phi_{\alpha\beta} + \left(\frac{1}{\bar{R}_\alpha}{}^h_0\theta + {}^h_1\theta\right){}_{(m+1)}^{h}\Phi_{\alpha\beta} + \left(\frac{1}{\bar{R}_\alpha}{}^h_1\theta + {}^h_2\theta\right){}_{(m+2)}^{h}\Phi_{\alpha\beta} + \frac{1}{\bar{R}_\alpha}{}^h_2\theta\,{}_{(m+3)}^{h}\Phi_{\alpha\beta} $$

$$ {}^hP = {}^h_0\theta\,{}^h_0\Phi_{33} + \left[{}^h_1\theta + \left(\frac{1}{R_1} + \frac{1}{R_2}\right){}^h_0\theta\right]{}^h_1\Phi_{33} + \left[{}^h_2\theta + \left(\frac{1}{R_1} + \frac{1}{R_2}\right){}^h_1\theta + \frac{1}{R_1R_2}{}^h_0\theta\right]{}^h_2\Phi_{33} $$
$$ + \left[\left(\frac{1}{R_1} + \frac{1}{R_2}\right){}^h_2\theta + \frac{1}{R_1R_2}{}^h_1\theta\right]{}^h_3\Phi_{33} + \frac{1}{R_1R_2}{}^h_2\theta\,{}^h_4\Phi_{33} $$

$$ {}^h_mT = {}^h_0\theta\,{}^h_m\Phi_{33} + \left[{}^h_1\theta + \left(\frac{1}{R_1} + \frac{1}{R_2}\right){}^h_0\theta\right]{}_{(m+1)}^{h}\Phi_{33} + \left[{}^h_2\theta + \left(\frac{1}{R_1} + \frac{1}{R_2}\right){}^h_1\theta + \frac{1}{R_1R_2}{}^h_0\theta\right]{}_{(m+2)}^{h}\Phi_{33} $$
$$ + \left[\left(\frac{1}{R_1} + \frac{1}{R_2}\right){}^h_2\theta + \frac{1}{R_1R_2}{}^h_1\theta\right]{}_{(m+3)}^{h}\Phi_{33} + \frac{1}{R_1R_2}{}^h_2\theta\,{}_{(m+4)}^{h}\Phi_{33} $$

$\bar{R}_1 = R_2$ and $\bar{R}_2 = R_1$. Subscripts α and β take on values 1 and 2 but there is no summation over repeated subscripts. The definition of ${}^h_m\Phi_{ij}$ is given in eqn (36).

lines of curvature coordinates. The derivation of the component form of the equations from the tensor form is lengthy but straightforward, and is given in Doxsee (1988).

The terms with subscripts appearing in Tables 1–10 are the physical components of the vectors and tensors introduced previously. For example, the displacement vector **U** has the representation

$$ \mathbf{U} = U_1\mathbf{t}_1 + U_2\mathbf{t}_2 + U_3\mathbf{n}. \tag{35} $$

Analogous results hold for tensors.

In Tables 3 and 6, commas denote differentiation with respect to a coordinate. In Table 8, summation over 1 and 2 is implied by the repeated Greek subscripts. In Tables 9 and 10, the terms ${}_mC_{ijkl}$, ${}^h_m\Phi_{ij}$, and ${}^t_m\Phi_{ij}$ appear and are defined by

$$ {}_mC_{ijkl} = \int_{-t/2}^{t/2} C_{ijkl}z^m\,\mathrm{d}z $$

$$ {}^h_m\Phi_{ij} = \int_{-t/2}^{t/2} {}^h\Phi_{ij}z^m\,\mathrm{d}z $$

$$ {}^t_m\Phi_{ij} = \int_{-t/2}^{t/2} {}^t\Phi_{ij}z^m\,z \tag{36} $$

where subscripts i, j, k, and l each take on the values 1, 2, and 3.

The elasticities listed in Table 9 are approximations of those defined in eqns (29). The components of the elasticity tensors ${}_m\mathbf{B}$ and ${}_m\mathbf{G}$ can be expressed as an infinite sum (or expansion) in powers of the small number t/R, where t is the thickness and R a representative radius of curvature of the shell (Naghdi, 1963). N_C is defined to be the power of (t/R) in the last term kept in this expansion. Table 9 gives the first three terms of this expansion, which correspond to $(t/R)^0$, $(t/R)^1$ and $(t/R)^2$. The definitions of the stress–moisture and stress–temperature resultants in Table 10 contain terms of order $(t/R)^0$ and $(t/R)^1$. This is acceptable as long as $N_C = 1$ or 2. However, if $N_C = 0$, then for consistency, only the terms of order $(t/R)^0$ should be included in the definitions of the stress–moisture and stress–temperature resultants. The same holds true for the definitions of the applied traction resultants (Tables 4 and 7) and the body force resultants (Table 5). Also, in the definitions of the stress–moisture and stress–temperature resultants (Table 10) constant, linear, and quadratic variation of temperature and moisture changes through the thickness of the shell have been included (i.e. $N_\theta = 2$ in eqns (26)).

The numbers N_β, N_η, N_θ, and N_C determine the accuracy of the shell theory. Numerical examples, which demonstrate the differences between shell theories based on various values of some of these numbers, are given in Doxsee and Springer (1989a).

CONCLUDING REMARKS

The equations developed in this paper may be used to analyze the hygrothermal behavior of laminated composite shells. A numerical procedure suitable for obtaining solutions for shells of revolution subjected to axisymmetric changes in temperature and moisture is presented in Doxsee and Springer (1989a). Assessments of the accuracy of the analysis and the numerical procedure are given in Doxsee and Springer (1989a,b).

Acknowledgments—This work was supported in part by Hughes Aircraft Company, Space and Communications Group, with Dr Stanley S. Chang acting as the project engineer, and by General Motors Research Laboratories, through a fellowship to the author.

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